PERPENDICULAR LINES AND DIAGONAL TRIPLES IN OLD BABYLONIAN SURVEYING

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Abstract

The tablet Si. 427 demonstrates that diagonal triples, or Pythagorean triples as they are now known, were used by Old Babylonian surveyors to construct perpendicular field boundaries accurately. This is the only known application of diagonal triples from this time, and one of the most complete examples of applied geometry from the ancient world.

1. Introduction

Mesopotamian scribes understood the numerical relation between the sides and diagonal of a rectangle, now called the Pythagorean Theorem.[1] But what purpose did this early understanding serve at the time? To answer this question, we step outside the usual genre of mathematical texts and instead look at how geometry was used in the administration of land.

The surface area of a field determines the volume of seed, water, or harvest and therefore the area of fields is naturally important to the administration of any agricultural society. The Mesopotamian interest in land measurement began in the earliest times, and this interest grew into a profession. By at least Ur III times, or perhaps earlier, professional surveyors would measure the area of a field by making a rough subdivision of the land into simple shapes resembling rectangles, right trapezoids, and right triangles. The surface area of the whole field would be found by adding or removing these simple parts. A schematic drawing of the subdivision, along with individual areas and a total are the characteristic features of these early administrative texts, which are called field plans (Dunham 1986: 32). A small number of field plans, mostly fragments from the Old Akkadian period, were published by François Thureau-Dangin.[2]

1. The author would like to thank the İstanbul Arkeoloji Müzeleri for their kind assistance and for permission to publish the photograph of Si. 427 here, Prof. Wayne Horowitz for his encouragement and help with the translation, Prof. Steve Shnider for our correspondence which helped formulate the steps of the conjectured reconstruction, and most of all Prof. Norman Wildberger for many insightful conversations. The P numbers refer to the online CDLI archive; these can be accessed with the URL http://cdli.ucla.edu/P######, by inserting the appropriate number.

2. These are plan fragments from Girsu, some of which concern fields, others describe buildings, mostly published by Thureau-Dangin 1897: pls. 24–26, nos. 63–75, mostly republished in Thureau-Dangin 1903: 66–68, nos. 145–160, some of which can be indirectly joined together (dated to Lagash II by Röllig 1980–83: 464); see also de Genouillac 1921, no. 9312 (P214838). For a possible Early Dynastic/Early Sargonic example, possibly from Adab, see Bartash 2017: 94–95 no. 27 (P253758).

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Our understanding of Ur III surveying is much more complete due to an abundance of field plans from this period. According to Liverani (1990: 148 w. n. 6) more than thirty field plans have come down to us from the Ur III period, and at least ten more have been made available in subsequent years.[3]

Very few field plans date from Old Babylonian (OB) times. This article analyses one such text, Si. 427 from Sippar, which immediately distinguishes itself on account of the accuracy of its measurements and grid-like horizontal and vertical lines. These perpendicular lines were almost certainly built from special rectangles, called diagonal triples, whose sides and diagonal satisfy the Pythagorean Theorem. The idea of using the sides the (3,4,5) right triangle to make perpendicular lines was outlined by the Roman surveyor Balbus (ca. first century CE), but the use of diagonal triples in Si. 427 is both older and more sophisticated.

In most cases it is "usually fairly easy to distinguish between 'real' house and field plans drawn up by working scribes and school mathematics problems" (Robson 2008: 63–64). Robson lists accurate measurements, information about the date, administrator, and circumstances of the recording as the distinguishing features of a real field plan. In contrast, mathematical exercises tend to be numerically simple, anonymous, and undated. According to Robson's criteria, Si. 427 is a real field plan, even if it is undated. However, this classification must carefully account for the presence of diagonal triples. Closer inspection reveals that the plan actually contains two different surveys of the same field, which indicates the presence of a single object that motivated two mathematical representations.

2. Surveying and Mathematical Traditions

The primary role of Ur III surveyors was to measure the area of land. Field plans from this period were "evidently made out shortly before the harvest, in order to prepare an assessment of the yield" (Liverani 1990: 155). Property boundaries appear to be already established (Stephens 1953: 2) and were secondary objects that were measured to determine the area of the field. Oblique boundaries, which were not used to determine area, were never measured.

The role of the surveyor grew over time to meet the needs of an increasingly complex society. By the OB period at least, surveyors were also responsible for resolving private boundary disputes. This is indicated by the following passage, where a young boy boasts of his scribal training (Robson 2008: 122): "When I go to divide a plot, I can divide it; when I go out to apportion a field, I can apportion the pieces, so that when wronged men have a quarrel I soothe their hearts."

The expanded role of the OB surveyor is further evidenced by occasional reference to oblique boundaries in OB field plans (Nemet-Nejat 1982: 18–19). Hence the OB surveyor was a mathematically literate professional responsible for the mensuration of surface areas and, possibly unlike their earlier counterparts, the establishment of boundaries.

These ancient surveyors measured land using reed and rope. But more than mere tools these were revered symbols of royal justice "representing fair mensuration of land amongst the people" (Robson 2008: 122). In addition to their simple tools, surveyors received extensive mathematical training that enabled them to measure land effectively. Like other mathematically literate professionals, surveyors made calculations using Sexagesimal Place-Value System (SPVS) numbers that are similar to our modern floating-point numbers. Here, we will write SPVS numbers as a sequence of digits between 0 and 59, separated by a comma "," to indicate a 60-fold decrease in magnitude or a semicolon ";" when context indicates a clear separation between the integer and fractional parts of the number. For example, 5;42,30 means $5 + 42 \times 60^{-1}$

3. See, most recently, Brunke 2005.

+ 30 × 60^{-2}. It should be noted that there is no OB symbol equivalent to either 0 or ; hence some interpretation is always implicit in this notation.

Length was represented by a SPVS number in units of nindan, which is about six meters. Area was represented by several different units and in this discussion the relevant large units are the bur, eše and iku, which are equal to eighteen hundred, six hundred, and one hundred square nindan respectively. The small area units are the ubu, diš, and sar, which are equal to fifty, twenty-five, and one square nindan respectively (Friberg 2009: 3). Units smaller than diš are not usually found in Ur III surveys but are common in those from the OB period.

There was no general concept of angle, but the more fundamental notion of perpendicularity was well understood. The Akkadian word *mutarrittum*, meaning "direction of the plumb line," was used as a metaphor for the perpendicular side of a shape (Høyrup 2002: 228–29). Perpendicularity is emphasized in geometric drawings (Høyrup 2002: 228) and is implicit in the procedures for the area of rectangles, right triangles, and right trapezoids. For this discussion, we emphasize that scribes understood the sides of a rectangle were perpendicular without any comprehension of angles (Gandz 1929: 452).

Rectangles were characterized by the "Diagonal Rule" that equated the sum of the square sides with the square diagonal, an ancestor to Pythagoras's Theorem concerning right triangles (Friberg 2007b: 449–50). The Diagonal Rule occasionally occurs in the context of a right triangle, such as BM 85196 exercise 9 (Melville 2004: 150–52). However, it is more often seen in the context of rectangles (Friberg 2007b: 449) such as MS 3971 §3 (Friberg 2007b: 252–53) and MS 3052 §2 (Friberg 2007b: 274–75). Of course, the best-known OB document involving the Diagonal Rule is Plimpton 322 (Neugebauer and Sachs 1945: 38–41).

In OB terminology a rectangle was associated with three measurements: a short side, a long side, and a diagonal. If these three measurements are all SPVS numbers, then we refer to them as a diagonal triple. The simplest diagonal triple is the (3,4,5) triple, and mathematical exercises involving a diagonal triple almost always involve a multiple of this triple such as (0;45, 1, 1;15) or (1, 1;20, 1;40); see Neugebauer and Sachs (1945: 53–55). The next simplest triples are (5, 12, 13) and (8, 15, 17), and these also appear in mathematical exercises such as MS 3971 §3c and §3b (Friberg 2007b: 252–53), as well as Str. 364 §6a and §4 (Friberg 2007a: 249). More complicated diagonal triples are mentioned only occasionally.

So Mesopotamian scribes knew certain configurations of numbers made a rectangle, and that the sides of a rectangle are perpendicular. But why were they interested in the geometry of rectangles? The aforementioned mathematical exercises offer us no answer. Contrast this with the understanding of rectangles found in the Indian Sulvasūtras, which may date back as far as 800 BCE. The Baudhāna sutra contains a statement of the Pythagorean Theorem (or more accurately the Diagonal Rule since it is stated in terms of rectangles) along with the triples (3,4,5),(5,12,13),(8,15,17),(7,24,25) and (12,35,37) for the explicit purpose of creating rectangular altars. Because this sutra contains both the theory and its application together, it is easy to see this as a practical text written by people who "were interested in geometrical truths only so far as they were of practical use" (Thibaut 1875: 6). We argue that OB surveyors likewise had a practical interest in diagonal triples.

3. *YOS* 1, 22 (P142061, fig. 1)

Our discussion of field plans begins with the Ur III text *YOS* 1, 22, from Umma, which is one of the best-known examples of the genre.[4]

4. CDLI no. P142061; edited by Hanson 1952 and Stephens 1953, with collations.

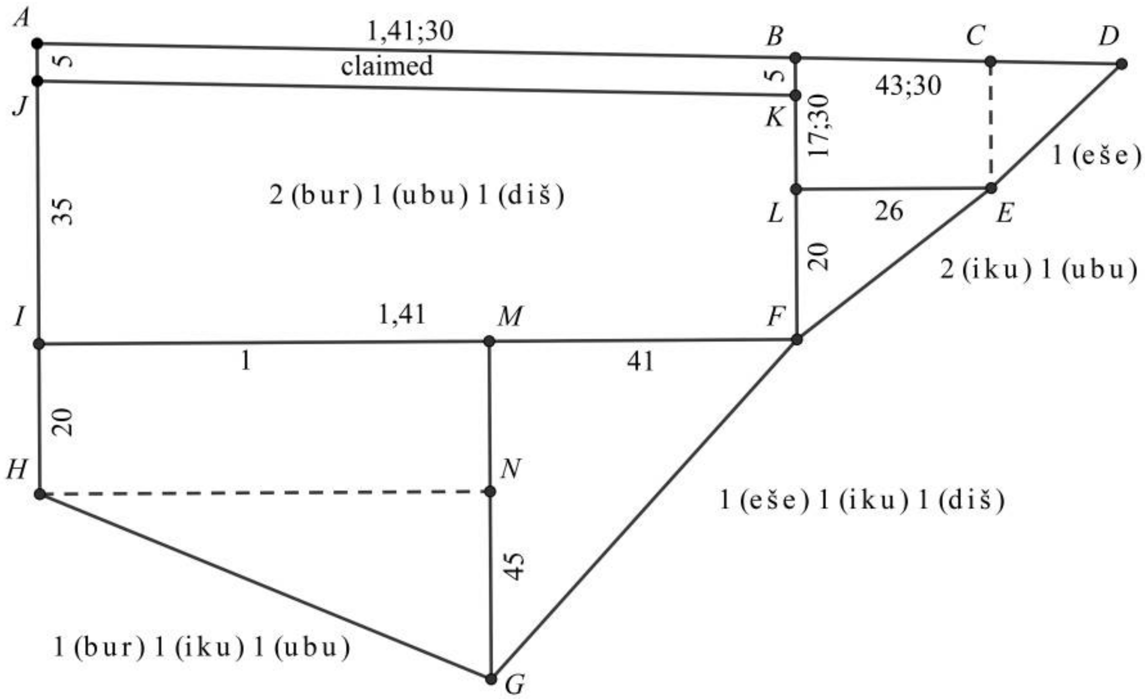

Fig. 1. Sketch of the field on *YOS* 1, 22, drawn to scale, after Robson (2008: 63). Dashed lines CE and HN added for consistency with Stephens 1953: 2, fig 2.

Figure 1 is a sketch of the field drawn on the obverse, which has been subdivided into shapes that are clearly intended to be rectangles, right triangles, and right trapezoids. However, the "rectangle" *JKFI* in *YOS* 1, 22 has opposite sides of similar but unequal length, and so it is not a genuine rectangle but a quasi-rectangle (Stephens 1953: 2). Such quasi-rectangles are found throughout Ur III field plans (Dunham 1986: 33). Evidently these early surveyors were unwilling or unable to construct perpendicular lines with enough accuracy to the make the opposite sides of a rectangle equal in length. While this is unsatisfactory form a theoretical perspective, it appears to have been acceptable for the purpose it was written. In terms of area, the quasi-rectangles provide a reasonable approximation to the true area (Høyrup 2002: 230), and since the Ur III surveyor's job was to approximate area, usually to the nearest diš, there was probably no need to ensure that opposite sides of a rectangle were anything more than approximately equal in length.

Table 1 lists theoretical areas based on the measurements in the sketch, adjacent to the areas inscribed near or within each shape (converted into sar). Because this is an Ur III survey, the inscribed areas have been rounded to the nearest diš. The area of *ABKJ* is explicitly excluded from the survey as it was claimed by another person, and there is one correction to be made. Throughout the Ur III to Late Babylonian periods the area of a quasi-rectangle was calculated by the *surveyor's rule*, which gives the area as the product of the average of opposite sides. By this rule the area of the quasi-rectangle *JKFI* should have been:

$$(35 + 32;30) \div 2 \times (1,41;30 + 1,41) \div 2 = 56,57;11,15.$$

But the inscribed area is equivalent to 1,01,15 sar. Stephens accounts for this as a calculation error where the scribe forgot to subtract 5 nindan from *FB* for the length of *FK* (Stephens 1953: 4).

The reverse contains the total of the inscribed areas along with additional information and a date. The complexity of the numbers involved, the additional information, and the date together indicate this is a real field plan and not a mathematical exercise (Robson 2008: 61–64).

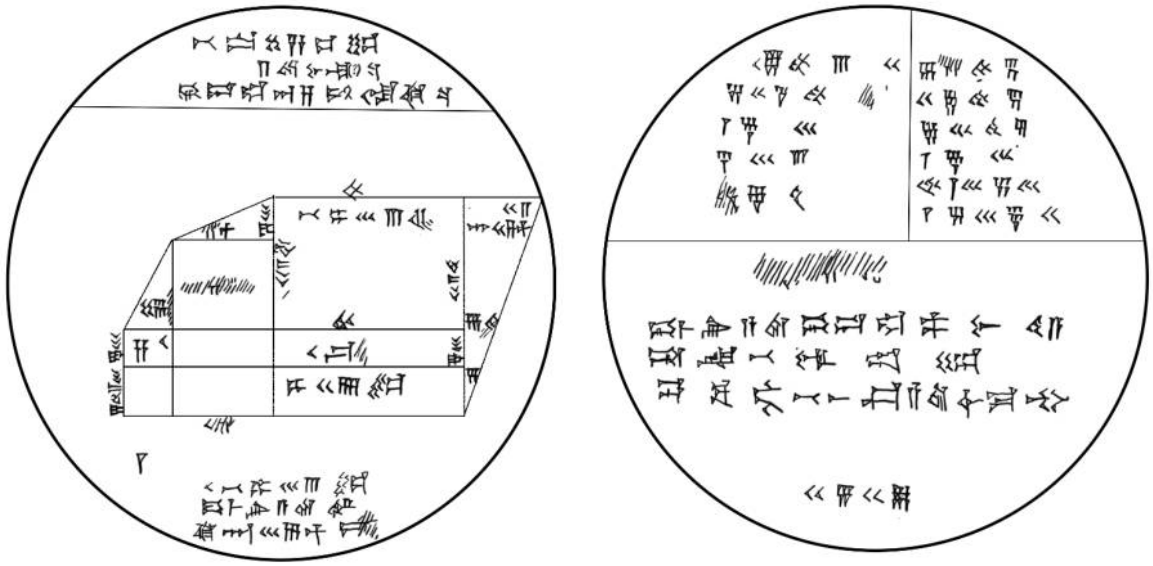

Fig. 2. Scheil's original hand copy of Si. 427 as traced by the author.

Table 1: Theoretical and inscribed areas in *YOS* 1, 22

Quasi-shape	Area (sar)		
	Theoretical		**Inscribed**
rectangle *JKFI*	(35 + 32;30) ÷ 2 × (1,41;30 + 1,41) ÷ 2 =	56,57;11,15	1,01,15
right trapezoid *IMGH*	1,00 × (20 + 45) ÷ 2 =	32,30	32,30
right triangle *MFG*	41 × 45 ÷ 2 =	15,22;30	15,25
right triangle *LEF*	26 × 20 ÷ 2 =	4,20	4,10
right trapezoid *BDEL*	17;30 × (26 + 43;30) ÷ 2 =	10,08;07,30	10,00

4. Si. 427 (P128359, figs. 2–4)

This is a lenticular tablet discovered in 1894 by the French expedition at Tell Abu Habba, ancient Sippar-Yahrurum, and deposited in the Musée impérial de Constantinople, now the İstanbul Arkeoloji Müzeleri, where it is archived with the catalog number Si. 427. It was first published in hand copy (fig. 2) with very partial edition by Scheil (1895: 33–34) and then cataloged as S. 427 in Scheil (1902: 134, without copy).[5]

Figure 4 contains photographs of the obverse and reverse of the tablet. The obverse contains a schematic diagram of a field (fig. 3) between two paragraphs of text and the reverse contains a table with two columns of numbers and text. There is damage to both sides, particularly to line 7 on the reverse, which

5. CDLI no. P128359

appears to have deteriorated since its original publication. Scheil's hand copy omits some important details that are clear from the photographs but is useful for restoring the part that has been damaged over the ensuing years.

Obverse Top

1.	1 (bur$_3$) GANA$_2$ 45 x sar	1 bur GANA 45 sar (x=erased sar)
2.	a.ša$_3$ sal.la-*tum*	of a marshy field[6]
3.	*qa*$_2$-*du-um* an.za.gar$_3$ *u*$_3$ kislah	together with the tower and threshing-floor.

Obverse Bottom

4.	1 (bur$_3$) 1 (eše) 4 (iku) 33 sar	1 bur 1 eše 4 iku 33 sar
5.	*ga-me-er* a.ša$_3$-*im*	total area of the field
6.	*u*$_3$ 3 (iku) ½ (iku) 36 ½ sar	and 3 iku ½ (iku) 36 ½ sar

Reverse Top

1.	18,53,20	6,3,55
2.	5,24,50	28,44
3.	1,7,30	8,30,45
4.	4,33	1,7,30
5.	[4]6,10	51,35,30
6.	…	1,5,37,30

Reverse Bottom

7.	…	
8.	*ga-me-er* a.ša$_3$ *qa*$_2$-*du-um* u$_2$.sal hi.a[7]	total (surface area) of the field, together with various marshes
9.	š*a* ^{d}en.zu-*be-el-ap-li*	of Sîn-bēl-apli
10.	*e-zu-ub* 1 (eše) 1 (iku) GANA$_2$ a.ša$_3$ *ši-ma-tim*	apart from 1 eše 1 iku field acquired through purchase.
11.	25,29	25,29

The transliteration and translation of obverse lines 2–3 and reverse bottom lines 8–10 are taken from a note attributed to Marten Stol (apud Veenhof 1973: 379). The term *ušallum* (u$_2$.sal), "water-meadows," is found in several OB texts and is taken to mean "the new accretion of land, along a watercourse, which has not yet been incorporated into the farm-land proper" (Veenhof 1973: 379). It can be translated as "lowlands" (*CAD* U, 296) or just "meadows." Line 7 is now mostly missing, but there appear to be the remains of numbers in Scheil's copy, perhaps a total.

Sîn-bêl-apli is a common Old Babylonian name well-attested at Sippar (Ferwerda and Woestenburg 1998: 264–65). This mildly suggests that Si. 427 is an OB text. Better evidence for this are the phrases "water-meadows" (Veenhof 1973: 367) and "tower and threshing floor" (Goddeeris 2002: 126) that occur in OB land-sale documents.

6. This reading follows Stol 1988: 174, who suggests that (u$_2$).sal.la in such contexts stands for Akkadian *raqqatum*, "marsh, swamp;" see now *CAD* R, 170 sub *raqqatu* B.

7. The line has been read from the hand copy. In its current state it reads: [*ga-me-e*]*r* a.ša$_3$ ⸢*qa*$_2$-*du-um*⸣ [u$_2$].⸢sal⸣ hi.a

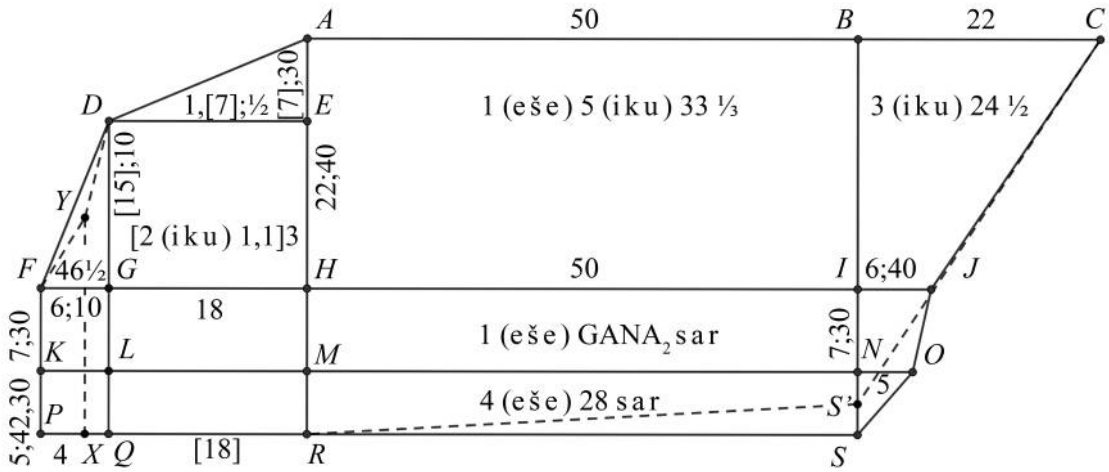

Fig. 3. Author's sketch of the field described in Si. 427, drawn to scale. Dashed lines FY, YD, XY, RS' and S'C added by the author.

In Scheil's analysis this is a "tablette cadastrale" (Scheil 1902: 134)—a conclusion initially suggested by the lenticular shape (Alivernini 2014: 3) and Robson's criteria for a real field plan since it contains sophisticated numbers, a named individual, and standard contractual phrases. However, the table of numbers on the reverse is unusual and suggests a mathematical aspect. This must have been apparent to Stol who categorized it as a "mathematical problem text" (Veenhof 1973: 379). If this is an exercise, then the extra information must be realia intended to give it a practical appearance (Robson 2008: 66). We shall return to the question of classification once the meaning of the numbers in the table has been established.

Table 2 lists the theoretical and inscribed areas for each shape. As usual the shapes are all rectangles, right triangles, and right trapezoids. Note that the parallel sides of the right trapezoids are horizontal, making this a "horizontal survey." Note also the equal opposite sides of the rectangles, which is highly unusual for a field plan.

Table 2: Theoretical and inscribed areas in Si. 427, together with Column 1.

	Area (sar)			
Shape	**Theoretical**		**Inscribed**	**Column 1**
rectangle *ABIH*	50 × 22;40 =	18,53;20	18,53 ⅓	18,53,20
right trapezoid *BCJI*	22;40 × (22 + 6;40) ÷ 2 =	5,24;53,20	5,24 ½	5,24,50
right triangle *ADE*	7;30 × 18 ÷ 2 =	1,07;30	1,07 ½	1,07,30
rectangle *DEHG*	18 × [15];10 =	4,33	4,33	4,33
right triangle *FGD*	6;10 × [15];10 ÷ 2 =	46;45,50	46 ½	[4]6,10
right trapezoid *JFKO*	7;30 × (1,20;50 + 1,19;10) ÷ 2 =	10,00	10,00	broken
right trapezoid *KOSP*	5;42,30 × (1,19;10 + 1,14;10) ÷ 2 =	7,17;38,20	7,08	broken

From table 2 it is immediately apparent that the numbers in column 1 concern horizontal survey areas written in SPVS form. The small difference between the theoretical and inscribed area of *BCJI* is due to the restrictions imposed by area units and to simplification. The area for *FGD* is damaged and appears to be 46,10, which is perhaps a calculation error. The area inscribed in *JFKO* is "10,00 $GANA_2$ sar" and exactly matches the theoretical area for this shape. The theoretical and inscribed areas for *KOSP* are different, which may be an error or a deliberate adjustment by the surveyor to better represent the area since it transpires that *KSOP* overestimates the actual land in question.

The obverse describes a marshy field with an area of 1 bur 45 sar. The first five consecutive entries of column 1 correspond to the region above the line *FJ*, which has a total area of

$$18,53;20 + 5,24;50 + 1,7;30 + 4,33 + 46;10 = 30,44;50 \approx 30;45 = 1 \text{ bur } 45 \text{ sar}.$$

Hence, we identify the line *FJ* as the marsh-boundary which delimits the marshy region from the rest of the field. The obverse also mentions an area of 3 iku ½ iku 36 ½ sar and because

$$1,7;30 + 4,33 + 46;10 = 6,26;40 \approx 6,26;30 = 3 \text{ iku } \tfrac{1}{2} \text{ iku } 36 \tfrac{1}{2} \text{ sar}$$

it is likely this is a reference to the marshy region left of the boundary line *AR*.

The correspondence established by table 2 allows us to restore the damaged entries, interpret the magnitude of these numbers, and suggests that the second column of numbers are also areas, but of a different kind since all inscribed areas have been accounted for.

Table 3: The table of numbers on the reverse, with totals and restorations based on theoretical area added by the author.

Column 1	Column 2
18,53;20	6,3;55
5,24;50	28,44
1,7;30	8,30;45
4,33	1,7;30
46;10	51;35,30
[10,00] [7,17;38,20]	1,5;37,30
Total area 48,02;28,20	Total area 46,23;23

Interpreting the second column as areas is difficult because only 1,7;30 explicitly appears in the sketch. However, the rectangle *DERQ* has the area

$$18 \times ([15];10 + 7;30 + 5;42,30) = 8,30;45$$

and combined with the right triangle *ADE* these consecutive entries make a vertical strip. This suggests that the numbers in the second column correspond to vertically aligned shapes and would also explain the presence of vertical lines in the lower half of the sketch, which are otherwise redundant.

The hand copy shows the rightmost field as a right triangle, and the area of the right triangle *BCS'* can be made to correspond to the first entry if we assume that *NS'* has a length of 2;55:

$$\tfrac{1}{2} \times (22;40 + 7;30 + 2;55) \times 22 = 6;3,55.$$

This assumption is justified by the fact that the area of the vertically aligned right trapezoid *ABS'R* now corresponds to the second entry:

$$\tfrac{1}{2} \times (35;52,30 + 33;05) \times 50 = 28,43;57,30 \approx 28,44.$$

Fig. 4. Si. 427, obverse and reverse

Incidentally, when drawn to scale the side *CS'* is in almost perfect alignment with *CJ*, which hints at the geometric reality underling the plan.

The remaining numbers can only be interpreted as 51;35,30 and 1,5;37,30 because placing the sexagesimal point elsewhere would make them either too small or large for areas. A tentative reconstruction of how the scribe arrived at these numbers is offered, although it is difficult to be confident. The first four entries in the second column correspond to vertically oriented shapes ordered from right to left. This pattern suggests 1,5;37,30 is the area of the leftmost shape, and a right trapezoid with vertical sides and area 51;35,30 should fill the space between. Since 4 is clearly written on the left side of the segment *PQ* and was not used in the horizontal survey, it might be the height of this right trapezoid. This is partially confirmed by the following calculations. If this hypothesized leftmost right trapezoid has area 1,5;37,30, height 4 and one side of length 7;30 + 5;42,30 = 13;12,30 then the other *XY* side must have length 19;36,15 because

$$\tfrac{1}{2} \times (13;12,30 + 19;36,15) \times 4 = 1,5;37,30.$$

On the other hand, if the remaining right trapezoid has an area of 51;35,30, a height 6;10 − 4 = 2;10 and one side is 15;10 + 7;30 + 5;42,30 = 28;22,30 in length, then the other side *XY* must have been 19;15 long because

$$\tfrac{1}{2} \times (19;15 + 28;22,30) \times 2;10 = 51;35,37,30 \approx 51;35,30.$$

There is a discrepancy here, but the two lengths 19;36,15 and 19;15 are strikingly similar. Our hypothesis is that the correct length for *XY* is 19;36,15 and that this was used for the first area calculation, but the middle digit (36) was omitted during the second area calculation.

Our analysis of the table on the reverse is summarized in table 3, along with the total area according to each survey. Our analysis of the table on the reverse is summarized in table 3, along with the total area according to each survey. Averaging differently oriented surveys is a technique that can be traced back to the Ur III period (Quillien 2003: 18; Brunke 2012: 47) and the fact that the average of the total area as measured by the horizontal and vertical surveys matches the area specified as the "full field."

$$\tfrac{1}{2} \times (48{,}02;28{,}20 + 46{,}23;23) = 47{,}12;55{,}40 \approx 47{,}13 = 1 \text{ bur } 1 \text{ eše } 4 \text{ iku } 33 \text{ sar}$$

further supports this interpretation of horizontal and vertical surveys.

Below the table of numbers there is a reference to a field acquired through purchase with an area of 1 eše 1 iku. Using consecutive numbers from the table on the reverse, but this time from the second column, we see this corresponds to the region left of the line AR:

$$8{,}30;45 + 1{,}07;30 + 51;35{,}30 + 1{,}05;37{,}30 = 11{,}35;28 \approx 11{,}40 = 1 \text{ eše } 1 \text{ iku}.$$

The complete narrative of Si. 427 can be deduced from this understanding of the words, numbers, and the locations of the fields. The tablet describes the consolidation of land with specific emphasis on marshy regions. The full field originally measured 47,13 sar, of which 30,45 sar was marshy. Approximately 11,40 sar of land was purchased, of which 6,26;30 sar was marshy. Line 7 on the reverse, traces of which are visible on Scheil's sketch but are now completely destroyed, was probably the area of the original field after the sale: 1 bur 3 iku 33 sar.

Overall, the purpose of the survey was to calculate areas and establish a new boundary. The horizontal survey was conducted because it matches the orientation of the original field, the vertical survey was conducted because it matches the orientation of the field acquired through purchase. In addition to making sense of the numbers, this interpretation explains the presence of legal terminology usually found in land sale documents.

The purchased field, measuring 7 iku, is a normal-sized private field (Liverani 1990: 158). The original field, measuring 28⅓ iku, is much larger and may have been owned by a wealthy individual. So much is certain: the surveyor went to extreme lengths to perform this subdivision as accurately as possible.

5. Perpendicularity

In what sense are the lines in Si. 427 perpendicular? The rectangles *ABIH* and *DEHG* have equal opposite sides and so more effort has been put into making the sides perpendicular compared with *YOS* 1, 22, for example. With this in mind we observe that the plan remarkably contains two or more likely three diagonal triples: the right triangle *ADE* (7;30, 18, 19;30) the rectangle *GHML* (7;30, 18, 19;30), and the rectangle adjacent to *FK* with measurements (4, 7;30, 8;30) although this last one is less certain because it is based on a reconstruction. Upon closer inspection, *GHML* is the (5, 12, 13) diagonal triple, scaled by a factor of two thirds, the right triangle *ADE* is exactly half *GHML*, and the rectangle adjacent to *FK* is the (8, 15, 17) triple, scaled by half.

The appearance of such simple and well-known diagonal triples, occurring in the same plan which appears also to contain accurately constructed perpendicular lines, cannot be dismissed as coincidence, moreso because the purpose of Si. 427 is to establish the very boundaries upon which these triples occur. Instead, the diagonal triples serve to explain how the perpendicular boundary lines were so accurately constructed.

Høyrup (2002: 230) is correct to remark that there are no truly perpendicular lines when it comes to surveying the real world. What we see here are perpendicular lines constructed to a high degree of accuracy. This accuracy is matched by the accuracy to which lengths are measured, such as 5;42,30, which is a whole order of magnitude more accurate than the lengths used in *YOS* 1, 22. This emphasis on accuracy is further reinforced by the fact that the total area was computed as the average of two separate surveys. Together, the accurate perpendicular lines, accurate measurements, and averaging all point to the conclusion that the individual who constructed Si. 427 wanted to make the most accurate calculations possible.

A diagonal triple is a special rectangle that is easy to construct and is useful in surveying because its sides can be extended into accurate perpendicular lines. The placement of the triple adjacent to *FK* suggests it was used to ensure the left boundary of the purchased field *FP* was perpendicular to the marsh-boundary *FJ*. Similarly, the placement of the triple *GHML* suggests it was used to ensure the right boundary of the purchased field *AR* was also perpendicular to the marsh-boundary *FJ*. The purpose of *ADE* is less clear. There appears to be little reason to place a right triangle here, but if we imagine that *ADE* was originally a rectangle then its sides would extend into the upper boundary of the original field *AC* and the right boundary of the purchased field *AR*, ensuring these important boundaries are perpendicular.

Presumably, the shape of the field determined the shape of *ADE*. As for the (8, 15, 17) triple, in the spirit of Høyrup's cut-and-paste geometry, it was probably only chosen because other known triples, such as the (3, 4, 5) or (5, 12, 13), were too thick or thin to fit within the available space.

One way in which the survey could have been conducted is as follows.

1. The surveyor began at the point *A*. To separate the two fields the surveyor built a rectangle with one side extending the upper boundary of the original field *AC*, another side perpendicular to *AC*, and diagonal which roughly matched the slope of the field as it curved away towards *D*.
2. This rectangle was then enlarged so that its diagonal matched length of the physical boundary/waterway along *AD*. At some point this rectangle was cut to form the right triangle *ADE*.
3. The side *AE* was extended by sight to the opposite boundary at *R*, this line is the new boundary that separates the purchased field from the remaining land.
4. The surveyor walked along the new boundary *AR* until the field was no longer marshy, then laid out the rectangle *GHML* and extended the side *GH* into marsh-boundary *FJ* by sight.
5. The left boundary of the purchased field *FP* could have been made by measuring out equal distances along the horizontal lines from *G* and *L*, or, more likely, by placing the (4, 7;30, 8;30) diagonal triple alongside *FK* and extending the side *FK* into *FP* by sight.

There is no indication as to how the other perpendicular lines were made. The presence of diagonal triples at the intersection of important boundaries is reasonable, but less significant intersections need not have received such close attention.

All horizontal lines in Si.427 are parallel by construction. This allows the surveyor to represent land using simple geometric shapes, but it causes problems because the upper and lower boundaries of the original field are not quite parallel. This is apparent when we contrast the two surveys: in the horizontal survey the lower boundary *RS* is necessarily horizontal because it is the side of a right-trapezoid; but in the vertical survey this same boundary is represented by an oblique segment *RS'*. Hence *RS*, which is necessarily horizontal by construction, is not a real boundary but an attempt to approximate an oblique boundary by a horizontal line.

We now return to the question of classifying Si. 427 as a mathematical exercise or real field plan. The text declares itself as a real field plan, however the presence of diagonal triples means that a higher standard of analysis is required. Whoever wrote Si. 427 applied diagonal triples to accurately construct perpendicular lines that subdivide an irregular shape. But was it a model geometry exercise or a survey of a real field? Mathematical exercises tended to have no geometric precision as "a consequence of [their] function in the specific context" of education (Høyrup 2002: 231). Exercises such as those found in Neugebauer and Sachs (1945: 44–48) were carefully constructed to be numerically simple rather than geometrically accurate. But Si. 427 features sophisticated arithmetic and exceptionally accurate geometry that is the exact opposite of the arithmetic and geometry found in mathematical exercises. So, despite its obvious mathematical aspects it bears none of the trappings of an exercise. On the other hand, the text itself declares its purpose concerns a marshy field split by purchase. The different measurements used in the horizontal and vertical surveys indicate an actual lower boundary that is closer to *RS'* than to *RS*, and similarly the actual upper left boundary is likely somewhere within the triangle *DFY*. This indicates the presence of an underlying object approximated by the horizontal and vertical surveys. In summary, the evidence unambiguously points to the conclusion that Si. 427 is exactly what it appears to be: a real field plan, but its strong mathematical aspects and focus on property boundaries mean that it would be more accurate to classify it as a *cadastral field plan that employs mathematical techniques*.

6. Conclusions

By OB times, surveyors were responsible for apportioning private land and Si. 427 shows that surveyors performed this role using geometric calculations that were exceptionally accurate for the time.

Establishing accurate perpendicular lines is a delicate task. Balbus outlined the use of the (3, 4, 5) triple along with two other geometric methods to construct perpendicular lines without specialized equipment. The theoretical nature of these other two geometric methods means they are not useful in practice and "it is not clear if they were applied in the field" (Lewis 2001: 22). But Si. 427 demonstrates that diagonal triples were actually used by surveyors to construct perpendicular lines. Moreover, the author of Si. 427 used a variety of diagonal triples that appear to have been chosen to match the shape of the field.

All of this confirms Adams's conjecture that the OB interest in diagonal triples "had something to do with the increasing need for cadastral accuracy" (Adams 2009: 6). Given the practical focus of Mesopotamian mathematics overall, few will be surprised to learn that diagonal triples served a practical purpose in the administration of land. But nobody could have hoped for such a clear example as Si. 427.

No analysis of diagonal triples would be complete without reference to Plimpton 322. There is a rich ongoing debate regarding the purpose of the Plimpton tablet, summarized by Britton, Proust, and Shnider (2011) with a later addition by Mansfield and Wildberger (2017). The debate so far has been heavily influenced by diagonal triples found in mathematical exercises such as MS 3971 §3 (Friberg 2007b: 252), MS 3052 §2 (p. 275) and Db_2 – 146 (Britton, Proust, and Shnider 2011: 549–50). But Si. 427 shows that diagonal triples were also practical objects used by surveyors, and this opens new directions for debate that are worth considering.

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